NORMAL CURVATURE BOUNDS VIA MEAN CURVATURE SMOOTHING

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ABSTRACT. Let $F_0 : M^n \to \overline{M}^{n+1}$ be a complete, immersed hypersurface with bounded second fundamental form in a complete ambient manifold with bounded geometry. Let $F_t : M^n \to \overline{M}^{n+1}$ be a solution to the mean curvature flow with initial data F_0. We show that the supremum and infimum of the normal curvature of the immersion F_t vary at a bounded rate. This is an analog of a result of Kapovitch on Ricci flow.

1. INTRODUCTION

In a recent paper [K] Kapovitch proved that the supremum and infimum of the sectional curvature of a complete manifold vary at a bounded rate under the Ricci flow, which generalized a result in Rong [R] to the noncompact case. In this short note, we will prove a mean curvature flow analog of this result. More precisely, we have the following

Theorem Let $F_0 : M^n \to \overline{M}^{n+1}$ be a complete, immersed hypersurface with bounded second fundamental form in a complete ambient manifold with bounded geometry. Let $F_t : M^n \to \overline{M}^{n+1}$ be a smooth solution to the mean curvature flow on $M^n \times [0,T]$ with initial data F_0. Then there exists a constant C depending only on n, T, the initial bound of the second fundamental form and the ambient manifold, such that $inf\kappa_0 - Ct \le \kappa_t \le sup\kappa_0 + Ct$, where κ_t is the normal curvature function of the immersion F_t.

(Here, as usual, by bounded geometry we mean that $\overline{M}^{n+1}$ has bounded injectivity radius and (norms of) covariant derivatives of the curvature tensor.)

Recall that the short time existence of the mean curvature flow in our situation is established by Ecker-Huisken [EH,Theorem 4.2]. (Actually Ecker-Huisken [EH] only consider the case $\overline{M}^{n+1} = R^{n+1}$, but their proof of Theorem 4.2 in [EH] can be easily adapted here, since in the general case one need only add some lower order terms in the evolution equation of the second fundamental form, which do not affect the original proof much.) Moreover from [EH] (cf. also [CY]) we know that the following estimates hold on $M^n \times [0,T]$:

$|\nabla^m A| \le \frac{C_m}{t^{(m+1)/2}}$,

$|H| \le C$,

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$|Rm_{g_t}| \leq C$, and
$|\frac{\partial g_t}{\partial t}| \leq C$.
(Here and below, we use C to denote various constants depending only on n, T, the initial bound of the second fundamental form and the ambient manifold.)

As in [K], the proof of our theorem is an application of the maximum principle combining with a technique of cut-off function, which is given in the second section.

2. Proof of Theorem

The proof is a modification of that of Kapovitch [K]. For simplicity we only consider the case $\overline{M}^{n+1} = R^{n+1}$, the general case can be treated similarly as remarked in the first section.

As in [K] we choose a nonnegative, nonincreasing, smooth function $\chi : R \to R$ with

(1) $\chi(s) = 1$ for $s \leq 1$, $\chi(s) = 0$ for $s \geq 2$,
(2) $|\frac{(\chi'(s))^2}{\chi(s)}| \leq 16$, and
(3) $|\chi''(s)| \leq 8$.

Given a point $z \in M$, let $d_z(x,t) = d_{g_t}(x,z)$ be the distance w.r.t. g_t, where g_t is the induced metric on M from the immersion F_t. Set $\xi_z(x,t) = \chi(d_z(x,t))$. Then as in [K] we have

(i) $0 \leq \xi_z \leq 1$, $|\frac{\partial \xi_z}{\partial t}| \leq C$,
(ii) $\frac{|\nabla \xi_z|^2}{|\xi_z|} \leq C$, and
(iii) $\triangle \xi_z \geq C$ in the barrier sense.

First we consider the case that $sup\kappa_t > 0$ for all $t \in [0,T]$. Let $\overline{A}(t) = sup\kappa_t$ and $\overline{A}_z(t) = max\{0, max_{\{(x,v)\}}\xi_z(x,t)\kappa_t(x,v)\}$, where x runs over M, and v runs over unit vectors (w.r.t. g_t) in T_xM. Of course $\overline{A}(t) = sup_{\{z\}}\overline{A}_z(t)$.

We wish to prove that the upper right-hand derivative of $\overline{A}_z(t)$ (which will be denoted by $\overline{A}_z{}'(t)$) satisfies $\overline{A}_z{}'(t) \leq C$ uniformly.

Let $\phi_z(x,v,t) = \xi_z(x,t)\kappa_t(x,v)$, by Hamilton [Ha,Lemma 3.5] we need only show that given $t_0 \in [0,T]$, $\frac{\partial \phi_z}{\partial t}(x_0,v_0,t_0) \leq C$ for any maximum point (x_0,v_0) of $\phi_z(\cdot,\cdot,t_0)$ such that $\kappa_{t_0}(x_0,v_0) > 0$.

Now we extend the vector v_0 by parallel translation along geodesics emanating radially out of x_0 w.r.t. g_{t_0}. Still denote this vector field by v_0.

Let $\Phi_z(x,t) = \xi_z(x,t)\kappa_t(x, \frac{v_0}{|v_0|_{g_t}}) = \xi_z(x,t)\frac{h_{ij}{v_0}^i{v_0}^j}{{|v_0|_{g_t}}^2}$, where h_{ij} is the second fundamental form of the immersion F_t, and ${v_0}^i$ is the i-th component of v_0 in (say) a normal coordinate system at x_0 w.r.t. g_{t_0}.

Using the evolution equation for the second fundamental form
$\frac{\partial}{\partial t}h_{ij} = \triangle h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} =: \triangle h_{ij} + P_{ij}$,
(cf. Huisken [Hu]) we compute
$\frac{\partial \Phi_z}{\partial t}(x_0,t_0)$
$= \triangle\Phi_z(x_0,t_0) - 2\nabla\xi_z \cdot \nabla h_{ij}{v_0}^i{v_0}^j(x_0,t_0) - h_{ij}{v_0}^i{v_0}^j\triangle\xi_z(x_0,t_0)$
$-\xi_z h_{ij}\triangle({v_0}^i{v_0}^j)(x_0,t_0) + \xi_z P_{ij}{v_0}^i{v_0}^j(x_0,t_0) + h_{ij}{v_0}^i{v_0}^j\frac{\partial \xi_z}{\partial t}(x_0,t_0)$
$+\xi_z h_{ij}{v_0}^i{v_0}^j\frac{\partial}{\partial t}(\frac{1}{{|v_0|_{g_t}}^2})(x_0,t_0)$.

Note $\triangle\Phi_z(x_0,t_0) \leq 0$, since $\Phi_z(\cdot,t_0)$ has a local maximum at x_0. Utilizing $\nabla\Phi_z(x_0,t_0) = 0$ and the property (ii) of ξ_z, we see that the second term in RHS is also bounded above. That the third term is bounded above is due to

$\kappa_{t_0}(x_0, v_0) > 0$ and the property (iii) of ξ_z. As in [R] we have $|\nabla^2 {v_0}^i|(x_0, t_0) \leq C$, $|\frac{\partial}{\partial t}|v_0|_{g_t}|(x_0, t_0) \leq C$, so the fourth term and the seventh term are also bounded. Finally that the fifth term and the sixth term are bounded follows trivially from property (i) of ξ_z (and the smoothing property of MCF). Then we obtain the desired estimate

$\frac{\partial \phi_z}{\partial t}(x_0, v_0, t_0) = \frac{\partial \Phi_z}{\partial t}(x_0, t_0) \leq C.$

It follows that $\overline{A}'(t) \leq C$.

As in [K], the general case can be easily reduced to this one by considering $\kappa_t + C$ instead, and the argument for $inf \kappa_t$ is similar.

Reference

[CY]B.-L. Chen, L. Yin, Uniqueness and pseudolocality theorems of the mean curvature flow, Comm. Anal. Geom. 15 (2007), no.3, 435-490.

[EH] K. Ecker, G. Huisken, Interior estimates for hypersurfaces moving by mean curvature, Invent. Math. 105 (1991), 547-569.

[Ha] R. Hamilton, Four-manifolds with positive curvature operator, J. Diff. Geom. 24 (1986), 153-179.

[Hu] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Diff. Geom. 20 (1984), no. 1, 237-266.

[K] V. Kapovitch, Curvature bounds via Ricci smoothing, Illinois J. Math. 49 (2005), no.1, 259-263.

[R] X. Rong, On the fundamental groups of manifolds of positive sectional curvature, Ann. Math. 143 (1996), no.2, 397-411.

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